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PROPERTIES OF THE GENERAL CONGRUENCY OF CURVES.

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IX.

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[Read June 26, 1899.]

TABLE OF CONTENTS.

ART.		PAGE
	Introductory,	363
1.	Vector expression for a congruency,	364
2.	Transformation from a system of parallel rays,	364
3.	Doubly infinite system of congruencies deduced,	364
4.	Having the same order, M,	365
5.	Correspondence of elements,	365
6.	Case of exception,	365
7.	Focal surface of the doubly infinite system,	366
8.	Depending on the Jacobian of the transformation,	366
9.	Correspondence of points on the focal surface,	367
10.	Theorems of contacts. A focal points on curve,	367
11.	Case of a volume element transforming into a line element,	368
12.	Not an ordinary singularity,	369
13.	Cuspidal curve on the focal surface,	369
14.	Notation for deriveds of second order,	369
15.	Locus of *Virtual Foci*,	370
16.	B virtual foci on a curve,	371
17.	Complex of tangents. *Rank* of congruency, R,	371
18.	Examples on parallel tangents,	372
19.	Osculating planes,	373
20.	*Class* of congruency, N,	373
21.	Properties of reciprocal congruency,	374
22.	Locus of points of consecutive osculations,	374
23.	Relations between the complex and the congruencies,	375
24.	Tangents in a plane,	375
25.	Pencils of a congruency,	376
26.	Curvatures of normal element,	376
27.	Special investigation involving ∇ and U_T,	376
28.	Concentration of a pencil,	377
29.	Density of a pencil,	377
30.	General investigation resumed,	378
31.	Expressions for density and concentration,	379
32.	Area of normal sections,	379
33.	Concentration is coefficient of contraction of area,	380
34.	Equation of boundary of pencil,	381
35.	Condition for orthogonalism,	382
36.	Special form of the condition,	383
37.	Other forms of the condition,	383
38.	Orthogonal surface, surface of zero density and locus of virtual foci,	383
39.	Third method of representing a congruency,	384
40.	Uses of this method,	385
41.	On the transformation producing an orthogonal congruency,	386
42.	Triad of mutually orthogonal congruencies,	386
43.	As transformed from mutually rectangular systems of lines,	387
44.	Case of pure strain of an element,	388
45.	Generalizations on transformations of congruencies,	388
46.	Congruencies of unicursal curves,	389
47.	Determination of the number A,	389
48.	On the class N,	389
49.	Surface enveloped by planes containing curves,	390
50.	Number B for unicursal curves,	390
51.	Special expressions for concentration and density,	391

INTRODUCTORY.

As far as possible Hamilton's well-known notation for linear and vector functions is retained in the present Paper. The symbol ϕ is used to denote a function of this kind. Its conjugate is denoted by ϕ'. The auxiliary functions ψ and ψ' as usual are defined by the equations

$$\psi V\lambda\mu = V\phi'\lambda\phi'\mu \quad \text{and} \quad \psi' V\lambda\mu = V\phi\lambda\phi\mu.$$

The symbolic cubic of ϕ is written in this form,

$$\phi^3 - m_1\phi^2 + m_2\phi - m_3 = 0.$$

In the frequently occurring equation $\rho = \theta\sigma$, θ is used as the symbol for the general vector function of the vector σ. The differential of this function is generally expressed by $\phi d\sigma$, though sometimes the notation $\theta_1 du + \theta_2 dv + \theta_3 dw$ is used. The notation $\phi_\iota d\sigma$ is frequently employed as an abbreviation for $Sd\sigma\nabla_\sigma S\iota\nabla_\sigma . \theta\sigma$, ι being a constant vector. The auxiliary functions ϕ'_ι, ψ_ι, and ψ'_ι correspond to ϕ_ι in the same way as the corresponding symbols without the *iota subscript* correspond to ϕ. The equation $\rho = \theta\sigma$ is interpreted as expressing the transformation of vectors σ into corresponding vectors ρ. Consequently the terms "the space (σ)" and "the space (ρ)" occasionally occur. The main drift of the Paper is the consideration of a congruency of curves as belonging to the doubly infinite system into which systems of parallel lines transform.

1. A congruency of curves of the most general type may be represented by means of a vector equation of the form

$$\rho = \chi(u, v, w),$$

in which two of the parameters, suppose u and v, serve to select a particular curve of the system, while the third parameter w specifies any particular point on the curve thus determined.

2. We may at once generalize this conception by supposing that u, v, and w are the components of a vector

$$\sigma = \alpha u + \beta v + \gamma w$$

along any three given vectors α, β, and γ, and we may write the equation of the congruency in the form

$$\rho = \theta\sigma,$$

and regard the curves of the congruency as transformations into the space (ρ) of a system of right lines in the space (σ) parallel to the direction of the vector γ.

3. It is obvious that the equation $\rho = \theta\sigma$ is capable of representing any one of a doubly infinite system of congruencies, each of which

answers to some particular direction of the vector γ. We shall see that all these congruencies have certain properties in common. They are all of the same order, and they have all the same focal surface.

4. In the first place they are all of the same order, M. In the equation $\rho = \theta\sigma$ to each value of ρ corresponds a certain definite number of vectors σ. This number is the number of points common to the three surfaces swept out by the vector σ, whose equations are

$$S\lambda\rho = S\lambda\theta\sigma,\ S\mu\rho = S\mu\theta\sigma,\ \text{and}\ S\nu\rho = S\nu\theta\sigma,$$

ρ, λ, μ, and ν being regarded as constant vectors.

If, through the extremity of each of the common vectors σ, a curve is drawn, these curves will transform into curves intersecting in the extremity of ρ. In particular, if parallel straight lines are drawn through the M points σ, they transform into M curves of the corresponding congruency intersecting in ρ.

5. To determine the focal surface it is necessary to trace the connexion between variations of ρ and the corresponding variations of σ. We shall write* in general

$$d\rho = d\theta\sigma = \phi d\sigma,$$

ϕ being a linear and vector function, variable from point to point inasmuch as the variable vector σ is involved in its constitution.

By means of this function we can compare the relation between points in the neighbourhood of ρ and points in the neighbourhood of the extremity of some one of the M vectors σ which correspond to ρ. To each of these vectors σ corresponds, of course, a different function ϕ.

6. In general the function ϕ admits of inversion by known methods, so that we have definitely $d\sigma = \phi^{-1}d\rho$ as well as $d\rho = \phi d\sigma$, or in other words, the relation between $d\rho$ and $d\sigma$ is, in general, one-to-one.

However, it may happen that for some vector τ

$$\phi\tau = 0$$

without having $\tau = 0$. In this case the component of $d\sigma$ which is parallel to τ is removed by the operator ϕ, so that, instead of unique correspondence between $d\rho$ and $d\sigma$, every vector in the plane of τ and $d\sigma$ whose

* In fact $d\rho = -Sd\sigma\nabla . \theta\sigma$ where ∇ refers to σ.

component at right angles to τ is the same as that of $d\sigma$ will correspond to $d\rho$. Thus, all curves through σ whose tangents are coplanar with τ and $d\sigma$, transform into curves having a common tangent $d\rho$ at the corresponding point ρ.

7. Interpreting the equation $\phi\tau = 0$, it is obvious that two consecutive points σ and $\sigma + d\sigma$ (where $d\sigma$ is parallel to τ) correspond to the same point ρ. Any curve whatever through σ and any other curve through $\sigma + d\sigma$, transform into curves intersecting in ρ. In particular, two parallel lines, one through the extremity of σ, and the other through that of $\sigma + d\sigma$, transform into curves intersecting in ρ. But consecutive curves of a congruency intersect at a focus, so the extremity of the vector ρ is a focus for every one of the doubly infinite system of congruencies of Article 3.

We can immediately find equations which determine the common locus of foci, or the common focal surface of all these congruencies. The condition to be satisfied is $\phi\tau = 0$. This requires m_3 to be zero where $m_3 S\alpha\beta\gamma = S\phi\alpha\phi\beta\phi\gamma$ is the third invariant* of the function ϕ. But m_3 is a function of the vector σ, so the surface $m_3 = 0$ in the region σ transforms into the *focal surface* represented by the equations

$$\rho = \theta\sigma \quad \text{and } m_3 = 0.$$

8. Since m_3 is an invariant, we have

$$Sd\rho d'\rho d''\rho = S\phi d\sigma\phi d'\sigma\phi d''\sigma = m_3 Sd\sigma d'\sigma d''\sigma,$$

where d, d', and d'' refer to three different variations of σ.

Hence, as is well known, we may write m_3 as a Jacobian in the form

$$m_3 = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

x, y, z being constituents of ρ, and u, v, w the corresponding constituents of σ.

* That is the coefficient m_3 of the symbolic cubic $\phi^3 - m_1\phi^2 + m_2\phi - m_3 = 0$.

9. It is obvious from the equation at the beginning of the last Article that an element of volume at σ is compressed into an element of a plane at the corresponding point; for $Sd\rho d'\rho d''\rho = 0$ requires $d\rho$, $d'\rho$ and $d''\rho$ to be coplanar. To put this in a clearer light, we may write the function ϕ in the trinomial form, $\phi\varpi = \lambda S\alpha\varpi + \mu S\beta\varpi + \nu S\gamma\varpi$. If, for some vector τ, $\phi\tau = 0$, it is obvious that $\lambda\mu$ and ν or $\alpha\beta$ and γ must be coplanar. In fact, $m_3 = S\lambda\mu\nu S\gamma\beta\alpha$. In this case* the function may (for new vectors $\lambda\mu$ and $\alpha\beta$) be reduced to the binomial form—

$$\phi\varpi = \lambda S\alpha\varpi + \mu S\beta\varpi \quad \text{and} \quad \phi V\alpha\beta = 0\,; \text{ so } \tau = V\alpha\beta.$$

Now, in general, if $d\rho = \phi d\sigma$ and $\delta\rho = \phi\delta\sigma$, we have

$$Vd\rho\delta\rho = V\phi d\sigma\phi\delta\sigma = \dagger\psi' Vd\sigma\delta\sigma,$$

so that, in general also, the relation is one-to-one between the directed elements of area $Vd\rho\delta\rho$ and $Vd\sigma\delta\sigma$. It is easily shown that for the trinomial form

$$\psi'\varpi = V\mu\nu S\gamma\beta\varpi + V\nu\lambda S\alpha\gamma\varpi + V\lambda\mu S\beta\alpha\varpi,$$

and for the binomial the function reduces to $\psi'\varpi = V\lambda\mu S\beta\alpha\varpi$. In this case $Vd\rho\delta\rho$ is parallel to a fixed vector $V\lambda\mu$. Surfaces through σ transform therefore into surfaces having a common tangent plane at ρ, except apparently in the case in which $\psi' Vd\sigma\delta\sigma = 0$. In that case $SV\beta\alpha Vd\sigma\delta\sigma = 0$, or $S\tau d\sigma\delta\sigma = 0$, or the vector τ of Art. 6 is tangent to the surface. As in that Article all the tangents $d\sigma$, $\delta\sigma$ are transformed into a common element of line at ρ, and the transformed surface has a cuspidal point with a cuspidal tangent in the common tangent plane.

It is hardly necessary to remark that this common plane touches the focal surface. This is obvious since the focal surface is transformed from a surface $m_3 = 0$ passing through σ.

10. It is apparent from the last Article that every surface in the region (σ) transforms into a surface touching the focal surface along a curve.‡

* "Elements of Quaternions," Art. 351, p. 497, new edition.

† ψ' is Hamilton's well-known function connected with ϕ by an equation as above.

‡ Analytically the conditions of contact are satisfied when the surface has a cusp with the cuspidal tangent touching the focal surface; this is the case of apparent exception of the last Article.

This curve is the transformation of that in which the original surface cuts $m_3 = 0$.

Again, any curve transforms into a curve touching the focal surface at the points corresponding to those in which the original cuts the surface $m_3 = 0$.

If A is the degree* (in σ) of the surface $m_3 = 0$, every curve derived from a right line touches the focal surface in A points. The number A is thus a common characteristic of the doubly infinite system of congruencies, and all their curves touch the focal surface in this number of points.

11. Having seen, in Article 7, that the focal surface corresponds to the case in which two of the M corresponding vectors (σ) of ρ approach coincidence, or to the case in which $m_3 = 0$, it is natural to inquire whether there is any singularity on the surface corresponding to the curve determined by putting both m_2 and m_3 equal to zero, or whether it is an ordinary singularity when three or more of the correspondents σ tend to coincide. It will appear that these two inquiries are quite distinct.

Returning to the binomial form of Article 9, we have $m_2 = SV\lambda\mu V\beta\alpha$. If this vanishes it may do so for $V\lambda\mu$ being at right angles to $V\beta\alpha$. The nature of the function $\psi'\varpi = V\lambda\mu S\beta\alpha\varpi$ is not essentially changed. The only peculiarity is that two of the vector solutions of $V\varpi\phi\varpi = 0$ coalesce into a single and determinate vector. It is quite different, however, if m_2 vanishes on account of $V\lambda\mu$ or $V\beta\alpha$ being zero. In that case ϕ reduces to the *monomial*,

$$\phi\varpi = \lambda S\alpha\varpi,$$

and the function ψ' vanishes identically. Every element $d\sigma$ is reduced to a single direction $d\rho$; every surface through σ transforms into a surface having a cusp at ρ. The solution of $\phi\tau = 0$ ceases to be determinate. The vector τ may be any vector whatever in a certain plane. This is the case in which three of the correspondents of ρ approach coincidence, and if three then an indefinite number, and there is an exception to the determinateness of the number M.

In this case it should be noticed that the symbolic cubic of the function

* Or at least the number of finite points in which a line meets the surface.

degrades into a quadratic, $\phi(\phi - m_1) = 0$. This is generally true when there is any indeterminateness of the axes. If two distinct solutions, ϖ_1 and ϖ_2, correspond to the same root g of any function whatever, the function $\phi - g$ removes the component of any vector ϖ in the plane of ϖ_1 and ϖ_2. If g' is another root $(\phi - g')(\phi - g)\,\varpi = 0$, or symbolically $(\phi - g')(\phi - g) = 0$.

12. The case in which ϕ reduces to a monomial is not an ordinary singularity when ϕ is not self-conjugate. For ϕ a monomial, $V\phi d\sigma\phi\delta\sigma = 0$, and every minor of the Jacobian of Article 8 must vanish. This involves four conditions. Corresponding to each of these is a surface in the region σ, and four surfaces have not in general a point in common, and the singularity is consequently not an ordinary singularity. Of course when it does occur, the surface $m_3 = 0$ has a double point.

When ϕ is self-conjugate only three conditions are necessary for the vanishing of every minor. In this case these special singularities exist.

13. There is, however, a cuspidal curve on the focal surface arising in a different manner. We have seen in Article 9 that if τ, or $\phi^{-1}0$, is tangent to a surface at σ, the transformed surface has a cusp at ρ. Let us, therefore, express that τ is tangent to the surface $m_3 = 0$. For the binomial form $\phi\varpi = \lambda S\alpha\varpi + \mu S\beta\varpi$, the function $\psi\varpi = V\beta\alpha S\lambda\mu\varpi$ is the conjugate of ψ' and $\psi\varpi$ is for all values of ϖ parallel to $V\beta\alpha$ or τ. If, then, ν is the normal to the surface $m_3 = 0$, the condition may be expressed by $S\nu\psi\varpi = 0$.

Hence $$\rho = \theta\sigma, \quad m_3 = 0, \quad \text{and } S\nu\psi\varpi = 0$$

determine a cuspidal curve on the focal surface, ϖ being any convenient vector.

The points of special singularity of the last article are of course, when they exist, situated upon this curve.

14. Up to the present (with a single exception in the last Article) we have been concerned only with the first deriveds of $\theta\sigma$. In general, for a double variation,

$$\delta d\rho = -\,\delta S d\sigma\nabla\,.\,\theta\sigma = -\,S\delta d\sigma\nabla\,.\,\theta\sigma + S\delta\sigma\nabla\,.\,Sd\sigma\nabla\,.\,\theta\sigma,$$

where ∇ operates on σ alone. For the most part we shall have to do only with cases in which $\delta d\sigma$ is zero. Moreover, as we shall have more especially to deal with a single congruency, because the properties to be dealt

with are not common to the whole system, it will be convenient to introduce a special notation. If the congruency is derived from lines originally parallel to ι, we shall write $\tau = \phi\iota$ as an abbreviation for the vector tangent to the curve at ρ which is transformed from the line parallel to ι at σ.

The vector tangent to a consecutive curve through $\rho + d\rho$ we shall represent by

$$\tau + d\tau = \phi\iota + \phi\, d\sigma,$$

$\phi_\iota d\sigma$ being the linear and vector function of $d\sigma$ represented by

$$\phi_\iota d\sigma = Sd\sigma\nabla S\iota\nabla \,.\, \theta\sigma.$$

This is readily proved to be true, since ultimately

$$\theta\,(\sigma + d\sigma) = \theta\sigma + \phi d\sigma,$$

and since we may regard $d\sigma$ as a constant. We have thus the simultaneous differentials $d\rho$, $d\tau$, and $d\sigma$ connected by the equations

$$d\rho = \phi d\sigma \quad \text{and} \quad d\tau = \phi_\iota d\sigma.$$

We shall occasionally write $d\rho = \phi\phi_\iota^{-1}d\tau$ and other analogous forms and we shall denote the cubic of $\phi\iota$ by

$$\phi_\iota^3 - m'_1\phi_\iota^2 + m'_2\phi_\iota - m'_3 = 0.$$

15. Any point on a line of a rectilinear congruency is generally the extremity of two lines of shortest distance between the given line and two consecutive lines. When the two lines of shortest distance coalesce, the point is said to be a virtual focus.

We can extend this conception to congruencies of curved lines. We shall express in the first place that the vector $d\rho$ joining two points ρ and $\rho + d\rho$ is at right angles to the tangents to the two consecutive curves at these two points. The conditions are obviously

$$S\tau d\rho = 0, \quad \text{and} \quad Sd\tau d\rho = 0,$$

or in terms of $d\sigma$, $\quad S\tau\phi d\sigma = 0, \quad \text{and} \quad S\phi_\iota d\sigma\phi d\sigma = 0.$

The first of these (regarded as a plane in $d\sigma$) determines two values of $d\sigma$ when combined with the second (regarded as a cone in $d\sigma$). To

express that these two values of $d\sigma$ coincide we have to express that the plane represented by $Sd\sigma\phi'\tau = 0$ touches the cone represented by

$$Sd\sigma\,(\phi'\phi_\iota + \phi'_\iota\phi)\,d\sigma = 0.*$$

This condition is well known to be

$$S\phi'\tau\,(\phi'\phi_\iota + \phi'_\iota\phi)^{-1}\,\phi'\tau = 0,$$

or in terms of ι, it takes the form

$$S\phi'\phi\iota\,(\phi'\phi_\iota + \phi'_\iota\phi)^{-1}\,\phi'\phi\iota = 0.$$

This condition, combined with $\rho = \theta\sigma$, gives the locus of virtual foci of the congruency. It will be developed later on (Art. 38), but it is worth while remarking here that as the function ϕ_ι is linear in the vector ι, the condition will become homogeneous and of the fourth order in ι when the function $\phi'\phi_\iota + \phi'_\iota\phi$ is inverted. This points to a radical distinction between this locus and the focal surface. For we may assume any point ρ, and write down the condition that it should be a virtual focus, and we get the equation of a cone of the fourth order in ι which determines a singly infinite system of congruencies which have the assumed point for a virtual focus. This locus depends, therefore, essentially on the particular congruency of the system under consideration.

16. To determine the number B of virtual foci upon any line of the congruency, we may, in the expanded condition of the last article, replace σ by $\kappa + x\iota$. The order of x in the resulting equation determines the number of virtual foci on the curve into which the right line $\sigma = \kappa + x\iota$ transforms. This number is evidently, in general, the same for every congruency of the system, and it must be regarded as a characteristic of the congruency, or rather of the system.

17. Closely connected with each congruency of the system is the complex of right lines which touch the curves of the congruency. The vector equation of the complex may be written in the form $\rho = \theta\sigma + t\phi\iota$ when t is a parameter. This is the vector equation of a triply infinite

* The function involved is written in this way in order to deal with the self-conjugate part only. This is obviously necessary from what follows.

system of right lines, involving, as it does, three independent parameters in σ.

We shall call R the order of the complex, that is the degree of the cone of rays whose vertex is any point, or the class of the curve enveloped by the rays which lie in any plane.

For the rays through the extremity of any fixed vector α we have the equation

$$V\phi\iota\,(\theta\sigma - \alpha) = 0,$$

which generally gives a curve locus for σ, and imposing the additional condition

$$S\lambda\phi\iota = 0,$$

where λ is given, we determine by these two conditions the number R of lines in a given plane and through a given point, or the number of tangents which can be drawn in a given plane and through a given point to the curves of the congruency.

We shall call this number the rank of the congruency. It is, in general, the same for every congruency of the system.

18. As a particular case of the results of the last article we have the equation

$$V\beta\phi\iota = 0$$

to determine the locus of points of contact of tangents to the curves which are parallel to a fixed direction β. This, coupled with $\rho = \theta\sigma$, specifies the curve in question.

If two neighbouring curves have, at consecutive points, parallel tangents, we must have

$$V\tau d\tau = 0, \quad \text{or} \quad V\phi\iota\phi_\iota d\sigma = 0.$$

Now, as in Art. 16, we may write $\sigma = \kappa + x\iota$, where κ is constant for each curve, but varies from curve to curve. Hence $d\sigma = d\kappa + \iota dx$. Eliminating dx from

$$V\phi\iota\phi_\iota(d\kappa + \iota dx) = 0$$

the resulting equation

$$S\phi\iota\phi_\iota\iota\phi_\iota d\kappa \quad = 0$$

determines a single consecutive curve satisfying the conditions. For we

may, without loss of generality, take $S\iota\kappa = 0$, so that $S\iota d\kappa = 0$. Hence we get, in general, a single value for $d\kappa$.

19. In the last article, we saw that the locus of points at which the curves have a given direction is a curve. We may write down many examples of such curve loci connected with a congruency, but that which appears to have most interest is the locus of points at which the binormals are parallel to a given direction.

If the plane $S\lambda\rho = 1$ osculates a curve at the point $\rho = \theta\sigma$,

$$S\lambda\theta = 1, \quad S\lambda\phi\iota = 0, \quad \text{and} \quad S\lambda\phi_\iota\iota = 0.$$

Or in full, if $$\sigma = \kappa + \iota x,$$

$$S\lambda\theta = 1, \quad S\lambda\frac{\partial\theta}{\partial x} = 0, \quad \text{and} \quad S\lambda\frac{\partial^2\theta}{\partial x^2} = 0.$$

Combining these equations we find at once

$$\lambda = \frac{V\phi\iota\,\phi_\iota\iota}{S\theta\,\phi\iota\,\phi_\iota\iota}$$

for the reciprocal of the vector-perpendicular to the osculating plane at the point $\theta\sigma$.

20. Now, this equation expresses a vector λ in terms of an auxiliary vector σ. We may regard it, therefore, as the equation of the congruency of curves traced out by the extremity of the variable vector λ.

The order of this new congruency, or the number of vectors σ which correspond to a given value of λ, determines the number of curves of the original congruency which an arbitrarily assumed plane osculates. This number may be called the Class N of the congruency of curves $\rho = \theta(\kappa + u\iota)$. It is generally the same for all the congruencies of the system $\rho = \theta\sigma$. It is obvious that there can be no corresponding characteristic for a rectilinear congruency, and that in the case of a congruency of plane curves, also, there is great speciality.

The curves of this congruency are, in a well-known sense, the reciprocals of the curves of the congruency $\rho = \theta(\kappa + u\iota)$. In fact, a curve of one congruency is the cuspidal edge of the developable surface into which the corresponding curve, on the other congruency, reciprocates.

21. As the focal surface of this new congruency corresponds to the points for which two consecutive values of σ correspond to the same value of λ, it is the locus of reciprocals of the vector perpendiculars on planes which osculate two consecutive curves of the congruency. Its reciprocal is the envelope of these planes.

22. It is easy to find equations which give the locus of points at which a plane can osculate two consecutive curves of a congruency.

In addition to the equations

$$S\lambda\theta = 1, \qquad S\lambda\phi\iota = 0, \quad \text{and} \quad S\lambda\phi_\iota\iota = 0,$$

we have, if the same plane osculates the consecutive curve

$$\rho = \theta(\kappa + u\iota) + \phi(d\kappa + \iota du)$$

at the point $\rho + d\rho$, on writing $d\sigma = d\kappa + \iota du$,

$$S\lambda\phi d\sigma = 0, \qquad S\lambda\phi_\iota d\sigma = 0, \quad \text{and} \quad S\lambda\phi_u d\sigma = 0,$$

where ϕ_u may be considered as an abridgment for $\frac{d\phi_\iota}{du}$. The third of these equations merely determines du, and may be omitted. The other two may be replaced by $S\lambda\phi d\kappa = 0$, and $S\lambda\phi_\iota d\kappa = 0$.

Eliminating λ and $d\kappa$ from these and from $S\iota d\kappa = 0$ we have at first on elimination of $d\kappa$, $S\iota\phi'\lambda\phi_\iota'\lambda = 0$, and on replacing λ by $V\phi\iota\phi_\iota\iota$, we find $S\iota\phi' V\phi\iota\phi_\iota\iota\phi_\iota' V\phi\iota\phi_\iota\iota = 0$. This condition, combined with $\rho = \theta\sigma$, determines the locus in question.

The condition may, however, be reduced to a simpler form. Since $\phi' V\phi\iota\phi_\iota\iota = V\iota\psi\phi_\iota\iota$, and $\phi_\iota' V\phi\iota\phi_\iota\iota = V\psi_\iota\phi\iota\iota$, we have, after easy reduction,

$$S\iota\psi\phi_\iota\iota\psi_\iota\phi\iota = 0.$$

This admits of the following interpretation. The product

$$\psi\phi_\iota \cdot \psi_\iota\phi = m_3 m_3'.$$

Hence it follows, that the condition expresses that ι is coplanar with a pair of axes of the function $\psi\phi_\iota$ or $\psi_\iota\phi$.*

* In fact, an equation $S\rho\phi\rho\psi\rho = 0$, or $S\rho\phi\rho\phi^2\rho = 0$, may be expressed by $(\phi^2 + x\phi + y)\rho = 0$; and it may be shown, that if ρ is not an axis, we must have $(\phi - g)(\phi - g')\rho = 0$, when g and g' are two latent roots.

23. It seems to be unnecessary to dwell with any great length on the relations between the original congruency $\rho = \theta\sigma$, the complex of tangents of Article 17, $\rho = \theta\sigma + t\phi\iota$, and the congruency of Article 20,

$$\lambda = \frac{V\phi\iota\phi_{,}\iota}{S\theta\phi\iota\phi_{,}\iota},$$

in all of which $\sigma = \kappa + u\iota$. The complex may be considered as consisting of the tangent developables of the curves of the original congruency, and each of these developables reciprocates into a curve of the second congruency. If each curve of the second congruency touches its focal surface in A' points, there are A' points on each corresponding curve of the original where the osculating plane osculates also a consecutive curve.

24. There are, however, two curves connected with the complex situated in any plane $S\lambda\rho = 1$, for which it may be worth while to find expressions. The first is the locus of points of contact of tangents in the plane. It is determined by $\rho = \theta\sigma$, $S\lambda\theta = 1$, and $S\lambda\phi\iota = 0$. The second is the envelope of tangents in the plane. If ω is a point on this envelope, we may differentiate $\omega = \theta\sigma + t\phi\iota$, leaving ω constant. This gives $\phi d\sigma + dt\phi\iota + t\phi_{,}d\sigma = 0$. These three vectors are, from the nature of the case, at right angles to λ.

Thus we may put $d\sigma = yV\phi'\lambda\phi'_{,}\lambda$. But $\phi d\sigma = yV\lambda\psi'\phi'_{,}\lambda$ and $\phi_{,}d\sigma = yV\psi'\phi'\lambda\lambda$ by a property of Hamilton's function ψ, so

$$V\lambda\psi'\phi'_{,}\lambda + dty^{-1}\phi\iota + tV\psi'_{,}\phi'\lambda\lambda = 0.$$

Operating by $V\phi\iota$, so as to eliminate $y^{-1}dt$, easy reduction gives

$$S\phi\iota\psi'\phi'_{,}\lambda = tS\phi\iota\psi'_{,}\phi'\lambda,$$

or $$t = \frac{m_3 S\lambda\phi_{,}\iota}{S\lambda\phi\psi_{,}\phi\iota}. \quad \text{Hence} \quad \rho = \theta\sigma + \frac{\phi\iota m_3 S\lambda\phi_{,}\iota}{S\lambda\phi\psi_{,}\phi\iota},$$

with the conditions $S\lambda\theta = 1$, $S\lambda\phi\iota = 0$, determines the envelope of tangents.

We see that this and the curve of contacts intersect in corresponding points only when $m_3 = 0$, or when $S\lambda\phi_{,}\iota = 0$. The first condition corresponds to points on the focal surface; the second to points of osculation. These conditions of intersection admit of immediate verification from geometrical considerations.

25. So far we have been with individual curves. We shall consider briefly some properties of small pencils of curves. About any point ρ we have M distinct small pencils answering to the M correspondents σ. In fact the pencils of lines parallel to ι through any small region near one of the points σ transforms into one of the pencils through the corresponding small region near ρ. It follows from the remark at the beginning of Article 11, that if ρ is on the focal surface, two of the pencils approach coincidence.

26. Confining the attention to one of the small pencils near ρ, imagine an element of a surface to be drawn through ρ orthogonal to each curve of the pencil. This conception is evidently possible. If K is one of the principal curvatures of the element, the vector to the corresponding centre of curvature is $\omega = \rho + K^{-1}U_\tau$, where $\tau = \phi\iota$* is normal to the element at ρ. Passing to a consecutive point on the line of curvature, ω and K remain unaltered. If then $D\rho$ is along the line of curvature,

$$D\rho + K^{-1}\, DU_\tau = 0.$$

27. In like manner, for the second line of curvature

$$D'\rho + K'^{-1}D'U_\tau = 0,$$

and from these

$$VD\rho D'\rho = K^{-1}K'^{-1}VDU_\tau D'U_\tau,$$

or

$$KK' = \frac{TVDU_\tau D'U_\tau}{TVD\rho D'\rho}.$$

The measure of curvature of the element, or the product of the two principal curvatures, is thus the ratio which the area determined on a unit sphere by lines parallel to the normals to the surface (or tangents to the curves) bears to the area of the element itself. This is the well-known expression for the *density* of the congruency at any point.

As the general equations in the notation of this paper may appear somewhat complicated, I shall for the moment suppose U_τ to be found or known as a function of ρ. This being assumed, we have

$$DU_\tau = -\, SD\rho\nabla.U_\tau = \phi_\rho D\rho,$$

* See Article 14.

ϕ_ρ being a new linear and vector function. Thus

$$\phi_\rho D\rho + KD\rho = 0, \text{ and } \phi_\rho D'\rho + K'D'\rho = 0;$$

so that $-K$ and $-K'$ are two of the roots of ϕ_ρ, and $D\rho$ and $D'\rho$ are the corresponding axes. It remains to find K and K' in terms of ϕ_ρ. Now, one root of this function is zero, or the function is *binomial*, because, for an arbitrary value of $d\rho$, $\phi_\rho d\rho = dU_\tau$ is at right angles to the given vector τ.* This being so, $K + K' = -m_1$ and $KK' = m_2$, m_1 and m_2 being the two invariants of the function. Since $\phi_\rho d\rho = -Sd\rho\nabla \,.\, U\tau$, we have evidently

$$m_1 = -S\nabla U\tau.$$

28. Hence, we deduce the general formula for the sum of the principal curvatures of any surface, or for double the mean curvature

$$K + K' = S\nabla U\tau,$$

or double the mean curvature is the "convergence" of the unit normal vectors. We may call this characteristic of a pencil of the congruency its *concentration.*

29. To deduce a similar expression for the measure of curvature of a surface, or for the density of a pencil of a congruency, we have for two vectors α and β

$$\psi'_\rho V\alpha\beta = V\phi_\rho\alpha\phi_\rho\beta = VS\alpha\nabla \,.\, U\tau S\beta\nabla' \,.\, U\tau',$$

in which we must drop the accents after the operations have been performed. This equation may be written

$$\psi'_\rho V\alpha\beta = -S\alpha\nabla S\beta\nabla' \cdot VU\tau U\tau' = -S\alpha\nabla' S\beta\nabla \,.\, VU\tau' U\tau,$$

or again

$$\psi'_\rho V\alpha\beta = -\tfrac{1}{2}(S\alpha\nabla \,.\, S\beta\nabla' - S\alpha\nabla' S\beta\nabla) \,.\, VU\tau U\tau'$$

$$= +\tfrac{1}{2} SV\alpha\beta V\nabla\nabla' \,.\, VU\tau U\tau'.$$

So that generally, for any vector γ,

$$\psi'_\rho\gamma = \tfrac{1}{2} S\gamma V\nabla\nabla' \,.\, VU\tau U\tau'.$$

* Compare Article 9.

This gives $m_2 = \frac{1}{2}SV\nabla\nabla' VU_\tau U_{\tau'}$; and we have the density or the measure of curvature expressed by $KK' = \frac{1}{2}SV\nabla\nabla' VU_\tau U_{\tau'}$.

30. Returning now to the notation of the present Paper, we have to transform the equation of Article 26,

$$KD\rho + DU_\tau = 0,$$

into a form involving $D\sigma$. Now $DU_\tau = D(\tau T\tau^{-1}) = T\tau^{-1}D\tau + \tau DT\tau^{-1}$, so that, for some scalar y, with whose value we are not concerned, the equation becomes

$$KT\tau D\rho + D\tau = y\tau,$$

or, as $\tau = \phi\iota, \quad D\rho = \phi D\sigma, \quad$ and $D\phi\iota = \phi_\iota D\sigma,$

$$KT\phi\iota\ \phi D\sigma + \phi_\iota D\sigma = y\phi\iota.$$

On inversion,

$$D\sigma = y(KT\phi\iota\,.\,\phi + \phi_\iota)^{-1}\phi\iota,$$

and because $S\tau D\rho = 0 = S\phi\iota\phi D\sigma = S\phi'\phi\iota D\sigma$, we have, at length, for the equation in K,

$$S\phi'\phi\iota(KT\phi\iota\,.\,\phi + \phi_\iota)^{-1}\phi\iota = 0.$$

It is now necessary to invert the function $KT\phi\iota\,.\,\phi + \phi_\iota$. For any two vectors α and β, we have, if M_3 is the third invariant of the function,

$$\begin{aligned} &M_3\,(KT\phi\iota\,.\,\phi + \phi_\iota)^{-1} V\alpha\beta \\ &\quad = V(KT\phi\iota\,.\,\phi'\alpha + \phi'_\iota\,\alpha)(KT\phi\iota\,.\,\phi'\beta + \phi'_\iota\,\beta) \\ &\quad = K^2T(\phi\iota)^2\,.\,V\phi'\alpha\phi'\beta + KT\phi\iota\ V(\phi'\alpha\phi'_\iota\,\beta + \phi'_\iota\,\alpha\phi'\beta) + V\phi'_\iota\,\alpha\phi'_\iota\,\beta. \end{aligned}$$

The only unusual part of this is the coefficient of K. We shall write $V(\phi'\alpha\phi'_\iota\,\beta + \phi'_\iota\,\alpha\phi'\beta) = \chi_\iota V\alpha\beta$, where χ_ι is a new linear vector function.*

Thus the equation for K is

$$S\phi'\phi\iota\left[K^2T(\phi\iota)^2\,.\,\psi\phi\iota + KT\phi\iota\,.\,\chi_\iota\phi\iota + \psi_\iota\phi\iota\right] = 0,$$

or,

$$K^2T(\phi\iota)^4 m_3 - KT\phi\iota S\phi\iota\phi\chi_\iota\phi\iota - S\phi\iota\phi\psi_\iota\phi\iota = 0.$$

* If we choose, χ_ι may be replaced by $\phi_\iota(\phi - m_1) + \phi(\phi_\iota - m'_1) + l_2$, where m'_1 is an invariant of ϕ_ι, and where the invariant l_2 is given by

$$l_2 S\alpha\beta\gamma = S\alpha(\phi'\beta\phi'_\iota\gamma + \phi'_\iota\beta\phi'\gamma) + S\beta(\phi'\gamma\phi'_\iota\alpha + \phi'_\iota\gamma\phi'\alpha) + S\gamma(\phi'\alpha\phi'_\iota\beta + \phi'_\iota\alpha\phi'\beta).$$

In Salmon's notation $\Sigma + K\Phi + k^2\Sigma' = 0$ for the reciprocal of the conic $S + kS = 0$, Φ depends on Σ and Σ', just as χ_ι depends on ψ and ψ_ι.

31. We have now for the concentration (Article 28),

$$K + K' = m_3^{-1} T\phi\iota S(\phi\iota)^{-1}\phi\chi_\iota(\phi\iota)^{-1},$$

and for the density,

$$KK' = -m_3^{-1} S(\phi\iota)^{-1}\phi\psi_\iota(\phi\iota)^{-1}.$$

Both of these become infinite on the focal surface where $m_3 = 0$. Combining either with $\rho = \theta\sigma$, we find surfaces of zero concentration and of zero density, and a curve along which both density and concentration are zero. Further, we see as the condition $S\phi\iota\phi\chi_\iota\phi\iota = 0$ is cubic in ι as occurring explicitly, we can select a singly infinite set of congruencies from the doubly infinite system by choosing for ι an edge of this cone, so that the concentration at any given point should be zero. In like manner, as $S\phi\iota\phi\psi_\iota\phi\iota = 0$ is a quartic in ι, we may select a singly infinite system, for which the density at a given point is zero, ι being now an edge of a quartic cone.

Again, if we put $\sigma = \kappa + u\iota$, and suppose u alone to vary, the conditions for zero concentration or density may be regarded as equations in u, so that, in this way again, we find a certain definite number of points on each curve where the concentration or the density is zero.

32. Before leaving the subject of pencils I shall give an expression for the area of the normal section of a pencil at any point. If $d\sigma$ and $\delta\sigma$ are at right angles to ι, they may be considered constant for any given pencil. Corresponding to these, we have, at any point of the pencil, $d\rho = \phi d\sigma$ and $\delta\rho = \phi\delta\sigma$. The components of these at right angles to $\phi\iota$ are $(\phi\iota)^{-1} V\phi\iota\phi d\sigma$ and $(\phi\iota)^{-1} V\phi\iota\phi\delta\sigma$, and the area of the cross-section is

$$TV(\phi\iota)^{-1} V\phi\iota\phi d\sigma \,.\, (\phi\iota)^{-1} V\phi\iota\phi\delta\sigma = \pm\, T(\phi\iota)^{-1} S\phi\iota\phi d\sigma\phi\delta\sigma = \pm\, T(\phi\iota)^{-1} m_3 S\iota d\sigma\delta\sigma$$

$$= \pm\, T(\phi\iota)^{-1} m_3 T\iota TV d\sigma\delta\sigma.$$

The area of the cross-section of a given pencil is thus proportional to

$$\frac{m_3}{T\phi\iota},$$

in which, as at the end of the last article, u is the only variable. We see that if $T\phi\iota$ does not vanish, the area of the cross-section vanishes at the focal surface where $m_3 = 0$.

33. It is not hard to see, that as we pass along the pencil, the coefficient of the rate of change of the area is equal to the concentration. If, for the moment, S is the area of the cross-section, this coefficient is

$$-S\,U d\rho \nabla_\rho \cdot \log S = T(d\rho)^{-1}\, d \log S = T(\phi\iota)^{-1} \cdot \frac{d}{du} \log S,$$

where u has the same meaning as at the end of the 31st Article. Now S varies as $m_3 T(\phi\iota)^{-1}$, so

$$\frac{d}{du} \log S = \frac{d}{du} \log m_3 - \frac{d}{du} \log T\phi\iota.$$

But $m_3 S\alpha\beta\gamma = S\phi\alpha\phi\beta\phi\gamma$, where α, β, and γ are any three fixed vectors, so

$$\frac{d}{du} \log S = \frac{S(\phi_\iota\alpha\phi\beta\phi\gamma + \phi\alpha\phi_\iota\beta\phi\gamma + \phi\alpha\phi\beta\phi_\iota\gamma)}{S\phi\alpha\phi\beta\phi\gamma} - S\frac{\phi_\iota\iota}{\phi\iota}.$$

$$= \frac{S(\phi_\iota\alpha\psi' V\beta\gamma + \phi\alpha\chi_\iota' V\beta\gamma)}{S\phi\alpha\phi\beta\phi\gamma} - S\frac{\phi_\iota\iota}{\phi\iota},$$

where χ_ι' is the conjugate of the function χ_ι of Article 30. But in this expression we may replace α, β, and γ by *any* other triad of vectors not necessarily fixed; for the term in which they enter is an *invariant* of the functions ϕ_ι and ϕ. We shall try if we can reduce the function to a single term. To do this, it is obvious on comparison that we should take $\alpha = \iota$, and $\psi' V\beta\gamma = \phi\iota$, or $m_3 V\beta\gamma = \phi'\phi\iota$, and, on making the substitution, the expression becomes

$$\frac{d}{du} \log S = \frac{S\phi_\iota\iota\phi\iota + m_3^{-1} S\phi\iota\chi_\iota'\phi'\phi\iota}{(\phi\iota)^2} - S\frac{\phi_\iota\iota}{\phi\iota};$$

so that, finally,

$$\frac{1}{T\phi\iota} \cdot \frac{d}{du} \log S = -\frac{S\phi\iota\chi_\iota'\phi'\phi\iota}{m_3 T(\phi\iota)^3} = -\frac{S\phi\iota\phi\chi_\iota\phi\iota}{m_3 T(\phi\iota)^3};$$

that is, by Article 31,

$$K + K' = -\frac{1}{T(\phi\iota)} \cdot \frac{d}{du} \log \frac{m_3}{T\phi\iota},$$

or, the *concentration* is equal to the coefficient of *contraction* of the cross-section of the pencil. This justifies anew the term concentration as applied to a small pencil of the congruency.

34. For the sake of definiteness, we may consider a pencil of the congruency obtained by transformation from the lines parallel to ι contained within a circular cylinder of small radius r. If we take ι, κ, and λ to be a unit and mutually rectangular system,

$$\sigma = \sigma_0 + r\kappa\cos v + r\lambda\sin v + u\iota,$$

is the equation of the small cylinder whose axis is $\sigma_0 + u\iota$. The equation of the bounding surface of the corresponding small pencil is

$$\rho = \theta(\sigma_0 + r\kappa\cos v + r\lambda\sin v + u\iota) = \theta(\sigma_0 + u\iota) + r\phi\kappa\cos v + r\phi\lambda\sin v.$$

The equation of a normal cross-section of the pencil is

$$\rho = \theta(\sigma_0 + u\iota) + r(\phi\iota)^{-1}V\phi\iota\phi\kappa\cos v + r(\phi\iota)^{-1}V\phi\iota\phi\lambda\sin v,$$

in which, of course, v is the variable. This is the equation of an ellipse, of which $r(\phi\iota)^{-1}V\phi\iota\phi\kappa$ and $r(\phi\iota)^{-1}V\phi\iota\phi\lambda$ are conjugate semi-diameters. The directions of the axis of the ellipse $\rho = \alpha\cos v_0 + \beta\sin v_0$ are found by expressing that this vector is at right angles to $\rho' = -\alpha\sin v_0 + \beta\cos v_0$. After a little reduction, we see that these directions are given by

$$S\,V\phi\iota\phi(\kappa\cos v_0 + \lambda\sin v_0)\,V\phi\iota\phi(\kappa\sin v_0 - \lambda\cos v_0) = 0;$$

or, more simply, by

$$S\psi'(\lambda\cos v_0 - \kappa\sin v_0)\psi'(\lambda\sin v_0 + \kappa\cos v_0) = 0;$$

or, finally, by

$$\tan 2v_0 = \frac{2\,S\psi'\lambda\psi'\kappa}{(\psi'\kappa)^2 - (\psi'\lambda)^2}.$$

This gives v_0 in terms of u, or the directions of the principal axes of the elliptic cross-section at each point of the pencil. For this value of v, the lengths of the principal semi-axes are

$$a = r\,T(\phi\iota)^{-1}T\psi'(\lambda\cos v_0 - \kappa\sin v_0), \text{ and } b = r\,T(\phi\iota)^{-1}T\psi'(\lambda\sin v_0 + \kappa\cos v_0).$$

35. We shall now consider the conditions under which it is possible to describe a surface orthogonal to the curves of the congruency. If $f\rho = 0$ is an orthogonal surface, and if τ is the tangent to a curve of the congruency through ρ, normal to the surface at that point, we must have $\tau = p\nabla_\rho f$, when p is some function of ρ. Then, if τ is expressed in terms of ρ,

$V\nabla_\rho\tau = V\nabla_\rho p \,.\, \nabla_\rho f$ and $S\tau\nabla_\rho\tau = 0$. If this function of ρ vanishes identically, it is possible to draw an infinite number of orthogonal surfaces; if it does not vanish identically, it is the equation of one or more surfaces orthogonal to the system of curves. We shall now translate this condition in ρ into the corresponding condition in σ. Symbolically

$$d = -Sd\rho\nabla_\rho = -Sd\sigma\nabla_\sigma = -S\phi d\sigma\nabla_\rho = -Sd\sigma\phi'\nabla_\rho,$$

and as this holds for all values of $d\sigma$, we have* $\nabla_\sigma = \phi'\nabla_\rho$. Hence, our condition becomes, since $\tau = \phi\iota$ and $\psi'\nabla_\sigma = m_3\nabla_\rho$,

$$S\phi\iota \,.\, \psi'\nabla_\sigma \,.\, \phi\iota = 0.$$

This, when it does not vanish identically, combined with $\rho = \theta\sigma$, determines the surface, or surfaces, orthogonal to the curves of the congruency.

36. If we suppose that the subscripts 1, 2, 3 applied to $\theta\sigma$ denote differentiation with respect to u, v, and w, where $\sigma = \iota u + \kappa v + \lambda w$, we have $\phi\iota = \theta_1$, and in general, $\phi d\sigma = -Sd\sigma\nabla_\sigma \,.\, \theta$

$$= \frac{Sd\sigma\kappa\lambda \,.\, \theta_1 + Sd\sigma\lambda\iota \,.\, \theta_2 + Sd\sigma\iota\kappa \,.\, \theta_3}{S\iota\kappa\lambda},$$

because, $\nabla_\sigma\iota \,.\, S\iota\kappa\lambda = V\kappa\lambda S\iota\nabla_\sigma \,.\, + V\lambda\iota S\kappa\nabla_\sigma \,.\, + V\iota\kappa S\lambda\nabla_\sigma \,.$

Also, $$\psi' d\sigma = \frac{V\theta_2\theta_3 S\iota d\sigma + V\theta_3\theta_1 S\kappa d\sigma + V\theta_1\theta_2 S\lambda d\sigma}{S\iota\kappa\lambda},$$

and, therefore, $$\psi'\nabla_\sigma \,.\, \phi\iota = \frac{V\theta_2\theta_3 \,.\, \theta_{11} + V\theta_3\theta_1 \,.\, \theta_{12} + V\theta_1\theta_2 \,.\, \theta_{13}}{S\iota\kappa\lambda},$$

so that the condition becomes

$$SV\theta_2\theta_3 V\theta_1\theta_{11} + SV\theta_3\theta_1 V\theta_1\theta_{12} + SV\theta_1\theta_2 V\theta_1\theta_{13} = 0.$$

This form is suitable for special cases.

37. For the general method of this Paper it is preferable to express the condition in terms of the functions ϕ and ϕ_ι. Since $Sd\sigma\nabla \,.\, \phi\iota = -\phi_\iota d\sigma$,

$$\psi'\nabla_\sigma \,.\, \phi\iota = \frac{\psi' V\kappa\lambda S\iota\nabla_\sigma + \psi' V\lambda\iota S\kappa\nabla_\sigma + \psi' V\iota\kappa S\lambda\nabla_\sigma}{S\iota\kappa\lambda} \,.\, \phi\iota$$

$$= \frac{\psi' V\kappa\lambda \,.\, \phi_\iota\iota + \psi' V\lambda\iota \,.\, \phi_\iota\kappa + \psi' V\iota\kappa \,.\, \phi_\iota\lambda}{S\iota\kappa\lambda}.$$

* Tait's *Quaternions*, Art. 480.

Now* $S\phi\iota \,.\, \psi' V\kappa\lambda \,.\, \phi_\iota\iota = S\phi'\phi\iota \,.\, V\kappa\lambda \,.\, \phi'\phi_\iota\iota$, and hence the condition may be expressed in the form,

$$S\phi'\phi\iota V(V\kappa\lambda \,.\, \phi'\phi_\iota\iota + V\lambda\iota \,.\, \phi'\phi_\iota\kappa + V\iota\kappa \,.\, \phi'\phi_\iota\lambda) = 0,$$

that is, $S\phi'\phi\iota\eta = 0$, or $S\phi\iota\phi\eta = 0$, where η is the spin-vector of the function $\phi'\phi_\iota$.

This expresses that the spin-vector of $\phi'\phi_\iota$, and the vector ι, transform into directions at right angles for points on the orthogonal surface.† It is, of course, satisfied when $\phi'\phi_\iota$ is self-conjugate.

38. There is an interesting relation between the orthogonal surface, the surface of zero density (Article 31), and the surface of Virtual Foci (Article 15). To find this in the simplest form we shall suppose the congruency to be given, as in Article 27, by $\tau = \chi\rho$, χ being a known function of ρ. We shall also use the notation‡ $d\tau = \phi_\rho d\rho$.

To find the virtual foci we have to express, as in the Article cited, that $S\tau d\rho = 0$ and $Sd\tau d\rho = 0$ lead to coincident values in $d\rho$. Hence, the "plane" $S\tau d\rho = 0$ must touch the "cone" $Sd\rho(\phi_\rho + \phi'_\rho)d\rho = 0$, or $S\tau(\phi_\rho + \phi'_\rho)^{-1}\tau = 0$, or again, if we write $\Phi_\rho = \frac{1}{2}(\phi_\rho + \phi'_\rho)$ for the self-conjugate part of ϕ_ρ, and Ψ_ρ related to Φ_ρ as ψ_ρ is related to ϕ_ρ, the equation of the locus of virtual foci is $S\tau\Psi_\rho\tau = 0$.

For the curvature of an orthogonal element, we have, as before, $D\rho + K^{-1}T\tau^{-1}\phi_\rho D\rho = y\tau$, and as $S\tau D\rho = 0$, $S\tau(KT\tau + \phi_\rho)^{-1}\tau = 0$, or on inversion,

$$K^2T\tau^4 - KT\tau S\tau(m'_1 - \phi_\rho)\tau - S\tau\psi_\rho\tau = 0.$$

Thus the surface of zero density is $S\tau\psi_\rho\tau = 0$.

Comparing the equations of these two loci we see the only difference is that, in the first, the non-conjugate part of ϕ_ρ has been removed. Now in full,

$$\begin{aligned}\Psi_\rho V\lambda\mu &= V\Phi_\rho\lambda\Phi_\rho\mu = V(\phi'_\rho + V\epsilon)\lambda \,.\, (\phi'_\rho + V\epsilon)\mu \\ &= \psi_\rho V\lambda\mu + V\phi'_\rho\lambda V\epsilon\mu + VV\epsilon\lambda\phi'_\rho\mu + VV\epsilon\lambda V\epsilon\mu,\end{aligned}$$

* In fact $S\phi\alpha\psi'\beta\phi_\iota\gamma = S\beta\psi V\phi_\iota\gamma\phi\alpha = S\beta\phi'\phi_\iota\gamma\phi'\phi\alpha$.

† A comparatively simple form is found when we introduce the function χ_ι of Art. 30.

‡ Here, for greater generality, τ is not supposed to be a unit vector. Consequently ϕ_ρ is not the same function as that in the article cited. For example, $\phi'_\rho\tau$ is not generally zero.

if ϵ is the spin-vector of ϕ_ρ. From this we easily see that, in general, for any vector, and in particular for τ,

$$\Psi_\rho\tau = \psi_\rho\tau + V\phi_\rho\epsilon\tau + \epsilon S\epsilon\tau.$$

Hence,

$$S\tau\Psi_\rho\tau = S\tau\psi_\rho\tau + (S\epsilon\tau)^2.$$

But we have already made use of the equation $\epsilon = \frac{1}{2}V\nabla_\rho\tau$, and by Article 35, $S\tau\nabla_\rho\tau = 0$ is the equation of the orthogonal surface whenever this expression does not vanish identically.

Hence, in general, the surfaces of zero density and of virtual foci touch one another along their intersection with the orthogonal surface. And when the congruency is orthogonal to an infinite number of surfaces, the surface of zero density coincides with the locus of virtual foci.*

Some difficulty may be avoided by remembering that density refers to *pencils*. At a point the density of one pencil may be zero, and that of another may be infinite.

39. There is yet another method of representing a congruency by means of a vector equation. We have seen, in Article 18, that the locus of the points of contact of parallel tangents is a curve. If the fixed direction of the tangents is $U\tau$, we may suppose $T\tau$ to be the variable parameter of the curve of contacts, and we may suppose that the vector equation of this curve is

$$\rho = \chi\tau,$$

in which $T\tau$ is the only variable. Now, allowing $U\tau$ to take all possible values, the same equation may be taken to represent the assembage of these curve loci, or in fact the congruency of these curves. But it also may be taken to represent the congruency of the touched curves, for to any given or assumed value of ρ corresponds a certain number of values of τ to be deduced from the equation $\rho = \chi\tau$, and these values of τ determine the directions of the curves of the congruency at the assumed point ρ.

* Of course in this ψ_ρ is supposed to be finite. It becomes infinite on the focal surface. See Article s. 31 and 40.

In accordance with the conception of transformation from one region to another, we see that the loci of points of contact of parallel tangents are to be regarded as transformed from a system of rays passing through a common origin in the region (τ).

Though this method of representation may appear artificial, it is a natural extension of a usual representation of a congruency of right lines by the equation $\rho = \chi(a) + ta$.

40. This method and that of Article 38 may both be called differential methods, as they deal primarily with the relation between the position of a point and the directions of the curves of the congruency through the point. Both are limited in their applications as the functions involved become infinite in certain cases.

We shall now write $\rho = \chi\tau$ and $d\rho = \phi_\tau d\tau$. By operations quite similar to those already employed we find for the equation in K,

$$S\tau\phi_\tau (\phi_\tau + K^{-1}T\tau^{-1})^{-1}\tau = 0.$$

This may be shown to reduce to*

$$K^2 m''_3 T\tau^4 + KT\tau S\tau (\psi_\tau - m''_2)\tau - S\tau\phi_\tau\tau = 0.$$

In this m''_1, m''_2, and m''_3 are the invariants of ϕ_τ and $m''_3 = 0$ on the focal surface.

We may deduce the relation $S\phi'_\tau\tau\Psi_\tau\phi'_\tau\tau = 0$ for the virtual foci, or reduced by a formula like that for Ψ_ρ of Article 38, we have

$$m''_3 S\tau\phi_\tau\tau + (S\tau\phi_\tau\epsilon')^2 = 0,$$

ϵ' being the new spin-vector.

For the intersection of this locus and the focal surface we have $m''_3 = 0$, and therefore $S\tau\phi_\tau\epsilon' = 0$.

The double factor shows that the focal surface and the locus of virtual foci touch along their common intersection with the orthogonal surface.

* Compare Article 38.

We see also that here the density is infinite, unless, indeed, the formula

$$KK' = -m''_3{}^{-1} S\tau^{-1}\phi_\tau\tau^{-1}$$

should become indeterminate.

These results should be compared with the concluding paragraphs of Article 38.

41. Returning to the condition of Article 35, that the curves of the congruency derived from the system of right lines parallel to ι should be orthogonal to a system of surfaces, we may, for brevity, write it in the form $S\iota\Phi\iota = 0$ where Φ is the determinate linear vector function involved in $S\phi\iota \,.\, \dot{\psi}\nabla_\sigma \,.\, \phi\iota = 0$. If we wish to ascertain whether any one congruency of the system possesses the orthogonal property, we see that the lines of the corresponding parallel system must be parallel to an edge of the cone $S\varpi\Phi\varpi = 0$; and, moreover, we must inquire whether the system of cones found by varying σ in Φ passes through a common line. For orthogonalism, the cone $S\varpi\Phi\varpi = 0$ must have an edge common to the three cones comprised in $\nabla_\sigma S\varpi\Phi\varpi = 0$. Now Φ involves differentials of $\rho = \theta\sigma$ of the second order. The conditions that the four cones should have a common line will consist, consequently, of a set of differential equations of the third order in ρ or $\theta\sigma$, this being the order of the differentials in $\nabla_\sigma S\varpi\Phi\varpi = 0$.

42. For the very important case in which the corresponding curves of three of the congruencies of the general system cut at right angles, it is easy to see, in various ways, that each of the three congruencies is orthogonal to a system of surfaces. In fact, the condition of Article 36, viz. :—

$$SV\theta_2\theta_3 V\theta_1\theta_{11} + SV\theta_3\theta_1 V\theta_1\theta_{12} + SV\theta_1\theta_2 V\theta_1\theta_{13} = 0,$$

reduces at once to

$$\theta_1{}^2 (S\theta_3\theta_{12} - S\theta_2\theta_{13}) = 0,$$

when

$$S\theta_2\theta_3 = S\theta_3\theta_1 = S\theta_1\theta_2 = 0.$$

Moreover, this condition is included in the conditions of the present article, as appears on differentation of the latter. In fact,

$$S\theta_1\theta_{23} = S\theta_2\theta_{31} = S\theta_3\theta_{12} = 0,$$

and the curves of two of the congruencies are lines of curvature on the system of surfaces orthogonal to the curves of the third.

43. In terms of the linear and vector function ϕ, we have $S\phi\kappa\phi\lambda = S\phi\lambda\phi\iota = S\phi\iota\phi\kappa = 0$, if the congruencies are transformed from systems of lines parallel to ι, κ, and λ, respectively. From the nature of these conditions it is evident that the condition for the focal surface $S\phi\iota\phi\kappa\phi\lambda = 0$ can be satisfied only by $T\phi\iota = 0$, $T\phi\kappa = 0$, or $T\phi\lambda = 0$. In this case, therefore, the real part of the focal surface will, in general, consist of isolated points, though it may in cases consist in part of real curves, or surfaces, if $\phi\iota$, for example, has a separable scalar factor, or vanishes with the vanishing of two scalar functions.

The conditions that the curves cut at right angles may be written in the form,

$$S\kappa\phi'\phi\lambda = S\lambda\phi'\phi\iota = S\iota\phi'\phi\kappa = 0.$$

Thus ι, κ, and λ are mutually conjugate with respect to the cone $S\varpi\phi'\phi\varpi = 0$. To find their directions when they exist, we have only to determine the triad self-conjugate to this cone and to $S\varpi d(\phi'\phi)\varpi = 0$ where $d(\phi'\phi)$ is the variation of the function corresponding to any variation $d\sigma$ of σ.

In this case the expressions of Articles 33 and 34 take very simple forms. Thus, the equation of the elliptic cross-section of a pencil derived from the lines contained within the *elliptic** cylinder

$$\sigma = \sigma_0 + u\iota + r\kappa\cos v + r'\lambda\sin v$$

is,

$$\rho = \theta(\sigma_0 + u\iota) + r\phi\kappa\cos v + r'\phi\lambda\sin v,$$

as $\phi\iota$, $\phi\kappa$, and $\phi\lambda$ are now mutually rectangular. Thus $r\phi\kappa$ and $r'\phi\lambda$ are the semi-axes major of the cross-section.

Hence, for congruencies of this class, if, at any point of an elliptic pencil the principal axes are along the lines of curvature of the orthogonal

* It must be remembered that here ι, κ, and λ do not, as in Article 34, form a mutually rectangular and unit system.

surface, the principal axes of any other normal section enjoy the same property.

Without loss of generality, we may suppose a preliminary strain effected in the region σ, so that ι, κ, and λ become mutually rectangular. We may, therefore, replace these vectors by i, j, and k. The conditions,

$$Sj\phi'\phi k = Sk\phi'\phi i = Si\phi'\phi j = 0,$$

show that i, j, and k are the fixed axes of the variable vector function $\phi'\phi$. This conception shows that we may write (if we did know it otherwise)

$$\phi\varpi = -\theta_1 Si\varpi - \theta_2 Sj\varpi - \theta_3 Sk\varpi$$

where θ_1, θ_2, and θ_3 are a set of variable vectors which remain constantly at right angles to one another. It would take too long to reduce the general formulæ of this Paper to their simplest forms for the type of congruency now under consideration, and, moreover, there is no difficulty in the matter.

44. For the special case in which the function ϕ, giving $d\rho$ in terms of $d\sigma$, is self-conjugate,* we must have $V\nabla_\sigma\theta\sigma = 0$, or $\theta\sigma = \nabla_\sigma f(\sigma)$, so that

$$\rho = \nabla_\sigma f(\sigma)$$

is the slope of a scalar function of σ.

Here $\phi\iota = -\nabla_\sigma S\iota\nabla_\sigma f$ is also the slope of a scalar function, and ϕ_ι, which is derived from $\phi\iota$ in the same way as ϕ is derived from θ, is also self-conjugate. We have seen, in Article 12, that, in this case, there are exceptional singularities on the focal surface.

45. It is, however, by no means necessary to confine our attention to the congruencies transformed by systems of parallel right lines in this region (σ). We may take any congruency in that region and transform, or we may write generally $\sigma = \theta_1\sigma_1$, and regard the region (σ) as transformed from a region σ_1. In this case

$$\rho = \theta(\theta_1\sigma_1), \quad \text{and} \quad d\rho = \phi d\,\theta_1\sigma_1 = \phi\phi_1\, d\sigma_1.$$

Now the m_3 invariant of $\phi\phi_1$ is $m_3 m'_3$, where m'_3 is the invariant of the

* See Article 12.

function ϕ_1. Hence we see that the focal surfaces of all congruencies derived in this way consist, in part, of the focal surface of the congruencies derived from systems of parallel right lines.

46. Before concluding, I shall give a few examples for the manageable case of congruencies consisting of unicursal curves. I write the equation of the congruency in the form

$$\rho = \frac{\psi w}{f w}$$

where ψ and f involve the parameter w algebraically, and in the n^{th} degree, and where the coefficients in these functions are functions of two variables, u and v, which serve for determining the individual curves. It will also be convenient to suppose ψ and f rendered homogeneous in two variables, w and w', w' being after the operations replaced by unity.

47. Comparing the two expressions,* $\rho = \theta\,(uvw) = f^{-1}\,.\,\psi$, we have $\theta_1 = (\psi_1 f - \psi f_1)\,f^{-2}$, &c., and for the focal surface the condition, $S\theta_1\theta_2\theta_3 = 0$, becomes

$$f^{-6} S\,(\psi_1 f - \psi f_1)\,(\psi_2 f - \psi f_2)\,(\psi_3 f - \psi f_3)$$
$$= f^{-4}[f S\psi_1\psi_2\psi_3 - S\psi\,(f_1\psi_2\psi_3 + f_2\psi_3\psi_1 + f_3\psi_1\psi_2)] = 0.$$

If ψ_4 and f_4 denote the differential coefficients of the fourth variable w', this reduces to

$$S(f_1\psi_2\psi_3\psi_4 - f_2\psi_1\psi_3\psi_4 + f_3\psi_1\psi_2\psi_4 - f_4\psi_1\psi_2\psi_3) = 0$$

when we suppose that f is not zero.†

Counting the order of w in this reduced expression, we find

$$A = 2\,(2n - 1)$$

for the number of points in which a curve touches the focal surface.‡ This gives the number of focal points on each curve for the unicursal case.

48. The class of the congruency N (Article 20) depends on the way in which u and v enter, and cannot be found until this is stated. It depends,

* See Article 36 for the notation.

† Of course, $f = 0$ corresponds to points at infinity.

‡ See Article 10. Darboux ("Leçons sur la Géométríe des Surfaces," Article 313) finds the number $n\,(n + 1)$ for *plane* curves.

in other words, on the arrangement of the curves as well as on their nature.

It is easy to see that this number is the degree in which w enters into the result of eliminating u and v from

$$S\lambda\psi_{33} = f_{33},\ S\lambda\psi_{34} = f_{34},\ \text{and}\ S\lambda\psi_{44} = f_{44},$$

$S\lambda\rho = 1$ being an osculating plane.

49. The equations of the last Article give

$$\lambda = \frac{f_{33} V\psi_{34}\psi_{44} + f_{34} V\psi_{44}\psi_{33} + f_{44} V\psi_{33}\psi_{34}}{S\psi_{33}\psi_{34}\psi_{44}}$$

for the equation of the reciprocal congruency of Article 21. As this is composed of unicursal curves of order 3 $(n-2)$, we find the number A' (of Article 23) of the points on each curve of the original congruency, where an osculating plane osculates a consecutive curve, to be $A' = 2(6n-7)$ by the formula of Article 47. However, we must have $n > 2$ when this formula can be used. For the case of a congruency of conics,

$$\rho = \frac{\alpha w^2 + 2\beta w + \gamma}{aw^2 + 2bw + c},$$

we have

$$\lambda = \frac{a V\beta\gamma + b V\gamma\alpha + c V\alpha\beta}{S\alpha\beta\gamma},$$

which represents no longer a congruency but the reciprocal of the surface enveloped by the planes of the conics. Similarly, for all plane curves we have a surface, not a congruency, and the number A' is inapplicable.

50. To find the number of virtual foci B (Article 16) on each unicursal curve, it is, perhaps, shorter to repeat the argument of Article 15 than to transform from the old to the new notation.

We have to express that

$$S\theta_3(\theta_1 du + \theta_2 dv + \theta_3 dw) = 0,$$

and

$$S(\theta_{31} du + \theta_{32} dv + \theta_{33} dw)(\theta_1 du + \theta_2 dv + \theta_3 dw) = 0$$

lead to a coincident pair of sets of values of du, dv, dw.

If in the notation of Article 46 we replace $\psi_1 f - \psi f_1$, $\psi_2 f - \psi f_2$, and $\psi_3 f - \psi f_3$ by λ, μ, and ν, the equations of the present article may obviously be replaced by

$$S\nu\,(\lambda du + \mu dv + \nu dw) = 0,$$

$$S\,(\nu_1 du + \nu_2 dv + \nu_3 dw)\,(\lambda du + \mu dv + \nu dw) = 0.$$

It is easy to see that λ and μ are of the order $2n - 1$ in w; ν, ν_1, and ν_2 of the order* $2n - 2$, and ν_3 of the order $2n - 3$. Regarding du, dv, and dw as trilinear coordinates, the number of virtual foci is the order of w in the condition that the first *line* represented by the first equation should touch the *conic* represented by the second, that is, the order of w in the determinant,

$$\begin{vmatrix} 2S\lambda\nu_1 & S\lambda\nu_2 + S\mu\nu_1 & S\lambda\nu_3 + S\nu\nu_1 & S\lambda\nu \\ S\mu\nu_1 + S\lambda\nu_2 & 2S\mu\nu_2 & S\mu\nu_3 + S\nu\nu_2 & S\mu\nu \\ S\nu\nu_1 + S\lambda\nu_3 & S\nu\nu_2 + S\mu\nu_3 & 2S\nu\nu_3 & \nu^2 \\ S\lambda\nu & S\mu\nu & \nu^2 & 0 \end{vmatrix} = 0.$$

The order of w sought is $16\,(n-1) + 2$, and if no further reduction is possible, this is the number of virtual foci.

This number, like the number A, depends on the nature of the curves, and not on their arrangement. The Rank of the congruency (Article 17), like the class, depends on the arrangement of the curves as well as on their nature

On the other hand, we find the equation of Article 36 to reduce in the notation of the present Article to

$$S\,V\mu\nu\,V\nu\nu_1 + S\,V\nu\lambda\,V\nu\nu_2 + S\,V\lambda\mu\,V\nu\nu_3 = 0,$$

and as $V\lambda\mu$, &c., have f as a factor, this becomes of the order $7\,(n-1)$ in w, so that a surface can be drawn orthogonal to the curves at $7\,(n-1)$ points on each.

51. In conclusion, for the concentration and density of a congruency of

* At least w divides off in the determinant in the case of ν.

unicursal curves, the curvatures of the normal element are to be obtained from the equation,

$$\theta_1 du + \theta_2 dv + \theta_3 dw + K^{-1} V(\theta_{31} du + \theta_{32} dv + \theta_{33} dw)\theta_3^{-1} U\theta = 0;$$

or

$$f^2(\lambda du + \mu dv + \nu dw) + K^{-1} V(\nu_1 du + \nu_2 dv + \nu_3 dw)\nu^{-1} U\nu = 0.$$

This gives the quadratic in K,

$$S(Kf^2\lambda + V\nu_1\nu^{-1}U\nu)(Kf^2\mu + V\nu_2\nu^{-1}U\nu)(Kf^2\nu + V\nu_3\nu^{-1}U\nu) = 0.$$

reducing to

$$K^2 f^2 S\lambda\mu\nu + Kf^2 T\nu^{-3}[S\lambda\mu\nu S\nu\nu_3 - \nu^2 S(\mu\nu\nu_1 + \nu\lambda\nu_2 + \lambda\mu\nu_3)]$$

$$- T\nu^{-4}(S\lambda\nu S\nu\nu_2\nu_3 + S\mu\nu S\nu\nu_3\nu_1 + S\nu\nu S\nu\nu_1\nu_2) = 0.$$

From this may be deduced the number of points on each curve at which the concentration or density is zero.